The Generalized Davis-Skodje Problem

Reference papers: S. Singh, J.M. Powers, and S. Paolucci, *J. Chem. Phys.* **117**,1482 (2002).

In order to test methods for generating infinite-dimensional slow manifolds, the Davis and Skodje model is generalized to include diffusion effects and defined in the domain $x \in [0, 1]$:

$$\frac{\partial y_1}{\partial t} = -y_1 + D_1 \frac{\partial^2 y_1}{\partial x^2}, \tag{1}$$

$$\frac{\partial y_2}{\partial t} = -\gamma y_2 + \frac{(\gamma - 1)y_1 + \gamma y_1^2}{(1 + y_1)^2} + D_2 \frac{\partial^2 y_2}{\partial x^2}, \tag{2}$$

with boundary conditions

$$y_1(t, 0) = y_2(t, 0) = 0 \qquad \text{and} \qquad y_1(t, 1) = a, \quad y_2(t, 1) = b. \tag{3}$$

where again $\gamma > 1$ gives a measure of stiffness for the system, $D_1 > 0$ and $D_2 > 0$ are diffusion coefficients, and a and b are arbitrary constants. If γ is increased, stiffness will increase. The equilibrium solution of the system is given by

$$y_1(x) = a \, \frac{\sinh(x/\sqrt{D_1})}{\sinh(1/\sqrt{D_1}}, \tag{4}$$

$$y_2(x) = b \, \frac{\sinh(x\sqrt{\gamma/D_2})}{\sinh(\sqrt{\gamma/D_2})} + \int_0^1 G(x, s) F(y_1(s)) \, ds, \tag{5}$$

where the Green's function $G(x, s)$ is given by

$$G(x, s) = \begin{cases} \frac{\sinh(\sqrt{\gamma/D_2}(s-1)) \sinh(\sqrt{\gamma/D_2}x)}{\sqrt{\gamma/D_2} \sinh(\sqrt{\gamma/D_2})}, & 0 \le x \le s, \\ \frac{\sinh(\sqrt{\gamma/D_2}(x-1)) \sinh(\sqrt{\gamma/D_2}s)}{\sqrt{\gamma/D_2} \sinh(\sqrt{\gamma/D_2})}, & s \le x \le 1, \end{cases} \tag{6}$$

and

$$F(y_1(x)) = -\frac{1}{D_2} \left(\frac{(\gamma - 1)y_1 + \gamma y_1^2}{(1 + y_1)^2} \right). \tag{7}$$

and is illustrated in Fig. 1. The basic problem is to efficiently and accurately compute an approximation to the slow invariant manifold embedded in the infinite-dimensional space. In addition, characterize the attractiveness of the slow manifold as a function of γ, D_1, D_2, a, and b, and given arbitrary initial conditions, provide an efficient strategy to determine approximately the time it takes for the solution trajectory to be sufficiently close to the slow manifold and the corresponding location near the manifold.

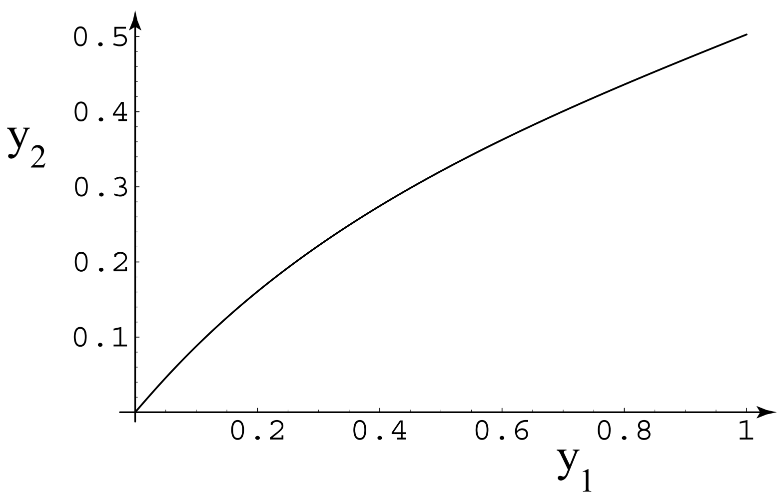

Figure 1: Equlibrium solution of (1) and (2) for $\gamma = 10$, $D_1 = D_2 = 0.1$, $a = 1$, and $b = \frac{1}{2} + \frac{1}{4\gamma(\gamma-1)}$.